Evaporation of a two-dimensional charged black hole

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Abstract

We construct a dilatonic two-dimensional model of a charged black hole. The classical solution is a static charged black hole, characterized by two parameters, m and q, representing the black hole's mass and charge. Then we study the semiclassical effects, and calculate the evaporation rate of both m and q, as a function of these two quantities. Analyzing this dynamical system, we find two qualitatively different regimes, depending on the electromagnetic coupling constant g_A. If the latter is greater than a certain critical value, the charge-to-mass ratio decays to zero upon evaporation. On the other hand, for g_A smaller than the critical value, the charge-to-mass ratio approaches a non-zero constant that depends on g_A but not on the initial values of m and q.

I. INTRODUCTION

Dilatonic two-dimensional models are very useful in studying various aspects of black holes (BHs). Callan, Giddings, Harvey and Strominger (CGHS) [1] introduced a dilatonic two-dimensional model to investigate the evaporation of a BH. They further used the two-dimensional model to explore the final stage of the evaporation process. The latter, more ambitious, goal of clarifying the endpoint of evaporation has proved to be illusive and hard to achieve [2]. Nevertheless, the two-dimensional model provided a useful description of the evaporation process in its initial phase, i.e., as long as the BH's mass is sufficiently large. Since then, many authors studied various aspects and various variants of CGHS's model.

The two-dimensional model studied by CGHS was that of an uncharged BH. The main objective of this paper is to develop an analogous model to describe the evaporation of an *electrically charged* two-dimensional BH. Our main motivation for studying charged BHs stems from the basic features of the classical, macroscopic, 4-dimensional BH solutions. The spacelike singularity of the Schwarzschild geometry disappears when an electric charge is added to the BH, and an inner horizon (IH) forms instead. Remarkably, the same situation occurs when an angular momentum is added to the BH (instead of an electric charge): In the Kerr solution, too, there is an inner horizon and no spacelike singularity. This observation is very relevant to reality, because realistic astrophysical BHs are believed to be rapidly spinning [3]. Spherically symmetric charged BHs thus provide a useful toy model for exploring various aspects related to the inner structure of the more realistic, spinning BHs (see e.g. Ref. [4]).

In the last decade several authors investigated two-dimensional models of charged BHs (see e.g. [5], [6], [7], [8]). McGuigan, Nappi, and Yost [5] studied such a classical model, with a dilaton coupling to the electric field. They considered a coupling $e^{-2\phi}$ of the dilaton field ϕ to the electromagnetic term in the action (see below). They found that, just as in the four-dimensional case, charged BHs admit an inner horizon instead of a spacelike singularity. However, their model does not include semiclassical effects, which are necessary for describing the BH's evaporation.

Later, Nojiri and Oda (NO) [6] considered a slightly modified dilatonic model, in which the dilaton coupling to the electric field is $e^{2\phi}$. In their model there is a large number N of chiral fermion fields (instead of the N scalar matter fields considered in Ref. [1]), which couple to both the curvature and the electromagnetic field. NO first studied the structure of the classical BH solution. Then, generalizing the method used by CGHS to the charged case, they added two effective semiclassical correction terms to the action, and derived from them the field equations at the semiclassical level. They have been able to solve some of the semiclassical field equations, which allowed them to analyze various aspects of the resultant semiclassical charged BH solutions.

Nojiri and Oda considered the dilaton coupling $e^{2\phi}$ rather than $e^{-2\phi}$, primarily because the former coupling makes some of the equations easier to solve. As it turns out, however, there is a remarkable difference between the two models, already at the classical level. The charged BHs with the coupling $e^{2\phi}$ do *not* admit an inner horizon [6] (instead they usually admit a spacelike singularity, just as in the uncharged case). Since our main motivation for considering charged BHs is to mimic the inner structure of the four-dimensional spinning BHs, we find it important to elaborate on the charged model with the dilaton coupling $e^{-2\phi}$. (Note also that $e^{-2\phi}$ is the coupling which emerges as the effective action in the low-energy limit of string theory. [6])

Motivated by these considerations, in this paper we start from the model developed by NO, and modify the dilaton coupling to the electromagnetic field from $e^{2\phi}$ to $e^{-2\phi}$. The resultant field equations are harder to solve at the semiclassical level. Nevertheless, it is possible to solve the equations describing the semiclassical effects in the adiabatic approximation, i.e. in the approximation where the background geometry is described by the static, non-evaporating, BH. [9] This approximation appears to be valid as long as the black hole is macroscopic, in which case the evaporation rate is small (namely, the relative change in the mass or charge during a dynamical time scale is $<< 1$). It is this macroscopic domain which will concern us throughout this paper. Thus, on the background of a classical static BH solution (with given mass and charge), we solve the semiclassical equations and derive from them the semiclassical fluxes of both energy-momentum and charge. This allows us to determine the evaporation rate of both the mass and charge of the BH.

We start in section II by writing the classical action and the corresponding field equations. We define new variables, which are used to simplify the equations and to present their general solution. This general solution is a two-parameter family of static black hole solutions, uniquely characterized by the two parameters m and q, which represent the BH's mass and charge. This classical solution was already given in Ref. [5], though in different coordinates.

Here we construct the classical solution in double-null coordinates, which are more suitable for the subsequent semiclassical calculations. We also extend the classical solution to include an outgoing (or ingoing) null fluid. (This extension is useful for describing the geometry of the evaporating BH at large distance from the horizon.)

In section III we consider the semiclassical effects. Following NO, we analyze the semiclassical effects by adding two effective correction terms to the classical action, expressed in terms of two new variables Z and Y (these variables describe the semiclassical fluxes of energy-momentum and charge, respectively). From this action we derive the semiclassical field equations. Then we solve the field equations for the two quantum variables Z and Y, assuming a background geometry of a static, classical BH. This solution yields an explicit expression for the fluxes of energy-momentum and charge, at any location (both outside and inside the BH), as a function of m and q. Based on these fluxes, in section IV we calculate the rate of evaporation of the mass and charge. We obtain a closed system of two first-order equations, describing $\dot{m}$ and $\dot{q}$ as functions of m and q, where an overdot denotes a derivative with respect to the external time. Both $\dot{m}$ and $\dot{q}$ are found to be negative, as one should expect. We then analyze this dynamical system, and find two qualitatively different regimes. If the electromagnetic coupling constant g_A is larger than a certain critical value, the charge-to-mass ratio decays to zero upon evaporation (as a certain power of the mass, which depends on g_A). On the other hand, for g_A smaller than the critical value, the charge-to-mass ratio approaches a non-zero constant, that depends on g_A but not on the initial values of m and q.

In section V we summarize and discuss our results. It should be emphasized that no attempt is made in this paper to investigated the final stage of evaporation. The goal here is to study the semiclassical evolution of the BH in the *macroscopic* phase, i.e. as long as the BH's mass is much larger than a certain mass. It is this phase in which the above mentioned adiabatic approximation is valid. In section V we further discuss this validity criterion for the adiabatic approximation, and find the range of mass values for which this approximation can be used.

II. CLASSICAL SOLUTIONS

A. field equations

We start with the classical action S_c given by NO [6], which includes an electromagnetic field coupled to charged matter represented by N left-handed chiral fermions:

$$S_c = \frac{1}{2\pi}\int d^2x\,\sqrt{-g}\{\mathrm{e}^{-2\phi}(R + 4(\nabla\phi)^2 + 4\lambda^2) \\ -\frac{\mathrm{e}^{a\phi}}{g_A^2}F^2 - \sum_{j=1}^{N} i\bar{\Psi}_j\gamma^\mu(D_\mu - iA_\mu)\Psi_j\}\ . \tag{1}$$

Here ϕ is a dilaton field, $\Psi_j = \binom{\psi_j}{0}$ are the N left-handed chiral fermion fields, g_A is the electromagnetic coupling constant, and D_μ denotes a covariant derivative. The Maxwell tensor $F_{\mu\nu}$ is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$. The coupling of the dilaton to the electromagnetic field term is written here, in a quite general form, as $\mathrm{e}^{a\phi}$. In Ref. [6], NO only analyzed the case $a = 2$. Here, for the reasons explained in the Introduction, we shall consider the case $a = -2$.

Following NO, we use the light-cone gauge for the electromagnetic field, namely

$$A_u = 0\ . \tag{2}$$

We also use double-null coordinates u, v, with

$$g_{uv} = -\frac{1}{2}\mathrm{e}^{2\rho}\ \ ,\ \ g_{uu} = g_{vv} = 0. \tag{3}$$

The action (1) (with $a = -2$) then reduces to

$$S_c = \frac{1}{2\pi}\int d^2x\,\{\mathrm{e}^{-2\phi}(4\rho_{,uv} - 8\phi_{,u}\phi_{,v} + 2\lambda^2\mathrm{e}^{2\rho}) \\ +\frac{4}{g_A^2}\mathrm{e}^{-2(\phi+\rho)}F_{uv}^2 + \frac{i}{2}\sum_{j=1}^{N}\psi_j^*\partial_v\psi_j\}\ , \tag{4}$$

where $F_{uv} = \partial_u A_v$. The Einstein equations are given by

$$0 = T_{vv} = \mathrm{e}^{-2\phi}(4\rho_{,v}\phi_{,v} - 2\phi_{,vv}) + \frac{i}{4}\sum_{j=1}^{N}(\psi_j^*\partial_v\psi_j - \partial_v\psi_j^*\psi_j)$$

$$+\frac{1}{2}A_v\sum_{j=1}^{N}\psi_j^*\psi_j \ , \tag{5}$$

$$0 = T_{uu} = \mathrm{e}^{-2\phi}(4\rho_{,u}\phi_{,u} - 2\phi_{,uu}) \ , \tag{6}$$

$$0 = T_{uv} = \mathrm{e}^{-2\phi}(2\phi_{,uv} - 4\phi_{,u}\phi_{,v} - \lambda^2\mathrm{e}^{2\rho}) + \frac{2}{g_A^2}\mathrm{e}^{-2(\phi+\rho)}F_{uv}^2 \ . \tag{7}$$

The dilaton equation of motion is

$$0 = -4\phi_{,uv} + 4\phi_{,u}\phi_{,v} + 2\rho_{,uv} + \lambda^2\mathrm{e}^{2\rho} + \frac{2}{g_A^2}\mathrm{e}^{-2\rho}F_{uv}^2 \ , \tag{8}$$

the Maxwell equations are

$$0 = \frac{8}{g_A^2}\partial_v(\mathrm{e}^{-2(\phi+\rho)}F_{uv}) + \frac{1}{2}\sum_{j=1}^{N}\psi_j^*\psi_j \ ,$$

$$0 = \frac{8}{g_A^2}\partial_u(\mathrm{e}^{-2(\phi+\rho)}F_{uv}) \ , \tag{9}$$

and the fermion fields satisfy

$$0 = \partial_u\psi_j \ . \tag{10}$$

We shall consider here solutions free of any classical matter. [1] That is, we shall only consider here the trivial solution

$$\psi_j = 0 \tag{11}$$

to Eq. (10). The Maxwell equations then reduce to

$$0 = \partial_u(\mathrm{e}^{-2(\phi+\rho)}F_{uv}) = \partial_v(\mathrm{e}^{-2(\phi+\rho)}F_{uv}) \ , \tag{12}$$

namely,

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[1] Presumably the charged BH was created by the collapse of the fermionic matter. Here, however, we are interested in the evaporation of the BH and not in its creation.

$$(\sqrt{2}/g_A)\ \mathrm{e}^{-2(\phi+\rho)} F_{uv} = \mathrm{const} \equiv \lambda q. \tag{13}$$

Substituting Eqs. (10) and (13) in the above system(5 -8), we obtain a simpler closed system, which includes the three Einstein equations

$$0 = T_{uv} = \mathrm{e}^{-2\phi}(2\phi_{,uv} - 4\phi_{,u}\phi_{,v} - \lambda^2 \mathrm{e}^{2\rho} + \lambda^2 q^2 \mathrm{e}^{4\phi+2\rho})\,, \tag{14}$$

$$0 = T_{vv} = \mathrm{e}^{-2\phi}(4\rho_{,v}\phi_{,v} - 2\phi_{,vv})\,, \tag{15}$$

$$0 = T_{uu} = \mathrm{e}^{-2\phi}(4\rho_{,u}\phi_{,u} - 2\phi_{,uu})\ , \tag{16}$$

and the dilaton equation

$$0 = -4\phi_{,uv} + 4\phi_{,u}\phi_{,v} + 2\rho_{,uv} + \lambda^2 \mathrm{e}^{2\rho} + \lambda^2 q^2 \mathrm{e}^{4\phi+2\rho}\,. \tag{17}$$

Note that the two equations (14) and (17) – to which we shall refer as the *evolution equations* – are hyperbolic, and are hence sufficient for determining the evolution of the two unknowns ρ and ϕ from prescribed initial data. [The other two equations (15 ,16) – the *constraint equation* – are consistent with the evolution equations: Any solution of the evolution equations whose initial data are consistent with the constraint equations, will also satisfy the constraints in the entire domain of dependence.]

B. New variables

To further simplify the analysis, we define the new variables

$$R = \mathrm{e}^{-2\phi}\ ,\ \ S = 2(\rho - \phi). \tag{18}$$

The evolution equations then reduce to

$$R_{,uv} = \lambda^2(q^2/R^2 - 1)\ \mathrm{e}^S, \tag{19}$$

$$S_{,uv} = (-2\lambda^2 q^2/R^3)\ \mathrm{e}^S. \tag{20}$$

The two constraint equations also take a simple form:

$$0 = T_{uu} = R_{,uu} - R_{,u} S_{,u} \ , \tag{21}$$

$$0 = T_{vv} = R_{,vv} - R_{,v} S_{,v} \ . \tag{22}$$

The two evolution equations (19,20) can be expressed in a compact form as

$$R_{,uv} = F(R)\ \mathrm{e}^S \quad , \quad S_{,uv} = F_{,R}\ \mathrm{e}^S \ . \tag{23}$$

where

$$F(R) = \lambda^2 (q^2/R^2 - 1) \ . \tag{24}$$

As it turns out, the system (23) is rather universal, as various general-relativistic models (e.g. several two-dimensional BH models, the three-dimensional BTZ model, and various spherically-symmetric four-dimensional models) satisfy the same form of hyperbolic system, with each model having its own function $F(R)$. For example, in the spherically-symmetric four-dimensional model of a charged BH with (or without) a cosmological constant, if one defines $R \equiv r^2$ and $\mathrm{e}^S \equiv r g_{uv}$ (where r is the area coordinate and u, v are two null coordinates), the two Einstein evolution equations take exactly the form (23) with

$$F(R) = aR^{1/2} + bR^{-1/2} + cR^{-3/2} \qquad (\mathrm{four-dimensional}) \ , \tag{25}$$

where a, b, c are constants (a and c represent the contributions of the cosmological constant and charge, respectively).

The non-linear hyperbolic system (23) [for a rather generic function $F(R)$] may serve as a useful mathematical toy model for studying various aspects of the theory of black holes, like gravitational collapse, singularity formation, and the no-hair principle. This, however, is beyond the scope of the present paper. Here we shall merely use the R, S variables to simplify the analysis, as the system (23) does not include first-order derivatives. We shall also use a few general features of this system – e.g. the form of its static black-hole solutions.

The generic solution of Eq. (23) does not necessarily satisfy the constraint equations (21,22). Of course, we are primarily interested here in the subclass of solutions which do satisfy the constraint equations, to which we shall refer as the *vacuum-like* solutions. Apart from the gauge freedom (i.e. the freedom to re-parametrize each of the two null coordinates u, v), this subclass is a one-parameter family of solutions [for a given $F(R)$], parametrized by the mass. These vacuum-like solutions are, in fact, the static black-hole solutions of the model. (For example, in the four-dimensional spherical electrovacuum case, these are the RN-deSittre family of solutions.) In the context of the specific model considered in this paper [with $F(R)$ of Eq. (24)], the vacuum-like solutions are nothing but the two-dimensional static electrovacuum solutions. We shall construct these static solutions, in double-null coordinates, in the next subsection.

One may also be interested in the wider class of solutions to the evolution equations (23), which do not necessarily satisfy the constraint equations (21,22). Such solutions may be interpreted as spacetimes perturbed by ingoing and/or outgoing null fluids, leading to non-vanishing contributions to T_{vv} and/or T_{uu}. We shall name such solutions as *radiative solutions*. For example, in the context of spherically-symmetric, four-dimensional, charged BHs, the mass-inflation solutions introduced in Ref. [4] belong to this class of radiative solutions [with the function $F(r)$ of Eq. (25)]. Note that although a radiative solution does not satisfy all the vacuum field equations, its evolution from Cauchy or characteristic initial data is completely determined from the (vacuum!) evolution equations, which form a closed hyperbolic system.

An important subclass of the radiative solutions are those which satisfy *one* of the constraint equations, but not the other one. Such solutions may be interpreted as spacetimes with a null fluid flowing in either the outgoing or ingoing direction. We shall refer to these solutions as the *Vaidya-like* solutions. For example, the geometry in the weak-filed region (i.e. at large R) of an evaporating BH can be well approximated by an outgoing Vaidya-like solution. In Appendix A we describe the construction of the general Vaidya-like solution in double-null coordinates.

C. The static black-hole solution

The general solution of the above system (19 - 22) (i.e. both the evolution and constraint equations) is a family of two-dimensional static, RN-like, black-hole solutions uniquely characterized by their mass and charge. This general solution was presented in Ref. [5] using Schwarzschild-like coordinates. For the analysis below we shall need the solution in double-null coordinates. We shall first describe the construction of this solution for a general function $F(R)$, and then restrict attention to our specific model, i.e. $F(R) = \lambda^2(q^2/R^2 - 1)$.

For a general function $F(R)$, we define

$$H(R) \equiv -\int^{R} F(R')dR' \,. \tag{26}$$

The static, vacuum-like, solution only depends on the spatial coordinate, which we denote x. We choose an Eddington-like gauge, such that $x = v - u$. The solution is then given by

$$\mathrm{e}^{S} = H(R) \;,\; R_{,x} = H(R) \;,\; x = v - u \qquad (H > 0). \tag{27}$$

From the above definitions of R and S, the metric function g_{uv} is given by

$$-2g_{uv} = e^{2\rho} = H/R \,. \tag{28}$$

Note that this Eddington-like solution is only valid in the region outside the BH where $H(R) > 0$ – this is the region which will primarily concern us in this paper. The solution exhibits a coordinate singularity whenever H vanishes [where g_{uv} vanishes, and so does $\det(g)$]. The lines $H = 0$ correspond to the horizons of the BH. These include the event horizon (EH), and [for functions $H(R)$ which admit more than one zero] also the inner horizon and/or the cosmological horizon. In the region inside the BH where $H(R)$ is negative, the solution can be expressed in a very similar form – see section III. Note that the solution includes one free parameter – the integration constant in Eq. (26) – which is related to the BH's mass (see below).

In the specific model considered in this paper, for which $F(R) = \lambda^2(q^2/R^2 - 1)$, we write $H(R)$ in the form

$$H(R) = \lambda^2(R - 2m + q^2/R) \tag{29}$$

(for notational convenience we take here the integration constant to be $-2m\lambda^2$). The vacuum-like solution is thus uniquely determined by the two parameters m and q, which are related to the black-hole's mass and charge, respectively. The root structure of the function $H(R)$ depends on the ratio between q and m. In this paper we shall consider non-extreme charged black-hole solutions, i.e. solutions with $m > q > 0$ (the restriction to positive rather than negative q does not cause any loss of generality). The equation $H(R) = 0$ then has two roots, at $R_\pm = m \pm \sqrt{m^2 - q^2}$, where R_+ corresponds to the EH and R_- corresponds to the IH. The function $H(R)$ is positive outside the BH, i.e. at $R > R_+$ (and also at $R < R_-$, but this range will not concern us in this paper), and negative between the two horizons. As was mentioned above, the solution (27) is only valid outside the BH, and a similar one, valid inside the BH, is given in section III.

The surface gravity κ_+ of the EH is defined by

$$\kappa_+ \equiv \frac{1}{2}(H_{,R})_{R_+} = \lambda^2(1 - q^2/R_+^2)/2\,. \tag{30}$$

For later convenience, we also express κ_+ in other useful forms:

$$\kappa_+ = \lambda^2(1 - m/R_+) = \lambda^2\left[(m^2 - q^2)^{1/2}/R_+\right]\,. \tag{31}$$

It is remarkable that in the two-dimensional case, unlike the situation in four-dimensional BHs, κ_+ only depends on q/m, and not on the BH's size. One explicitly finds

$$\kappa_+ = \lambda^2\left[1 - \left(1 + \sqrt{1 - (q/m)^2}\right)^{-1}\right]\,. \tag{32}$$

Note that κ_+ is a decreasing function of q/m, and for $0 \le q/m \le 1$ it takes the values $0 \le \kappa_+ \le \lambda^2/2$.

From Eq. (28), the metric function g_{uv} is given by

$$-2g_{uv} = \lambda^2(1 - 2m/R + q^2/R^2)\,. \tag{33}$$

At large R, this becomes $-2g_{uv} = \lambda^2$. It is useful to introduce new null coordinates

$$\tilde{u} = \lambda u,\ \tilde{v} = \lambda v\,, \tag{34}$$

such that

$$-2g_{\tilde{u}\tilde{v}} = 1 - 2m/R + q^2/R^2\,, \tag{35}$$

which yields the desired asymptotic behavior, $-2g_{\tilde{u}\tilde{v}} = 1$, at large R (this is also the type of gauge used by CGHS in the uncharged case). We shall refer to the $\tilde{u}, \tilde{v}$ coordinates as the *asymptotically-flat double-null coordinates.*

Since $R \equiv \mathrm{e}^{-2\phi}$ is dimensionless, the two parameters m and q are dimensionless too. These two parameters are proportional to the physical mass and charge of the BH, and we may refer to them as the dimensionless mass and charge, respectively. [One of the motivations for this association is the similarity of the metric function $g_{\tilde{u}\tilde{v}}$ in Eq. (35) to its counterpart in the standard four-dimensional Reissner-Nordstrom solution.] Throughout this paper we shall often refer to m and q simply as the BH's mass and charge (with some abuse of the standard terminology). To avoid confusion, we shall denote the dimensionful, physical, mass and charge of the BH by M and Q, respectively. The physical mass M, which is the total energy content of the system, is encoded in the asymptotic behavior of the metric tensor at large distance. Since this large distance corresponds to the limit $R \to \infty$, the physical mass will not be affected by the term q^2/R^2 in the metric (35). Hence M must be a function of m and λ solely. We can deduce $M(m, \lambda)$ from the form of the mass parameter M in the uncharged case, studied by CGHS. Comparing the static solutions of the two models, one finds [2]

$$M = 2\lambda m\,. \tag{36}$$

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[2]To relate M in the CHGS model to our notation, we can compare the value of the dilaton field at the EH. In our notation we find at the EH (for $q = 0$) $\mathrm{e}^{-2\phi} \equiv R = 2m$, whereas in CGHS we find at the EH $\mathrm{e}^{-2\phi} = M/\lambda$; cf. Eq. (11) therein.

The relation between q and Q may be revealed by comparing our static solution to that given (in different coordinates) in Ref. [5]. This comparison shows that Q is proportional to λq , in analogy with Eq. (36). We have not clarified the constant relating Q to λq (however, the explicit expression for Q is not required for the analysis below).

III. SEMICLASSICAL CORRECTIONS

A. Semiclassical field equations

In order to study the evaporation of the BH, we must consider the semiclassical contribution to the energy-momentum tensor and to the electromagnetic current. CGHS [1] showed that the semiclassical contributions can be treated by adding an effective correction term to the classical action. Nojiri and Oda [6] extended this method to the charged case (they also modified the action by considering fermionic rather than scalar matter fields). They found that the semiclassical contributions coming from the conformal anomaly can be represented by adding two correction terms to the classical action S_c:

$$S = S_c + S_\rho + S_\chi \ , \tag{37}$$

where

$$S_\rho = \frac{1}{2\pi} \int d^2x \, \sqrt{-g}\{-\frac{1}{2}(\nabla Z)^2 + \sqrt{\frac{N}{48}} Z \, R\} \tag{38}$$

and

$$S_\chi = \frac{1}{2\pi} \int d^2x \, \{-\frac{1}{2}\sqrt{-g}(\nabla Y)^2 + \sqrt{\frac{N}{2}} Y \epsilon^{\mu\nu} F_{\mu\nu}\} \ . \tag{39}$$

The term S_ρ, which emerges from the trace anomaly, contributes to the energy-momentum tensor. The second term S_χ comes from the chiral anomaly, and it contributes to both the electromagnetic current and the energy-momentum. Note the presence of two new variables, Z and Y, in the correction terms. These two degrees of freedom were introduced by NO in order to allow the representation of the semiclassical effects by local correction terms [6].

Expressing the metric in the double-null form (3) and using the electromagnetic gauge (2), the modified Einstein equations become

$$0 = T_{vv} = \mathrm{e}^{-2\phi}(4\rho_{,v}\phi_{,v} - 2\phi_{,vv}) + \hat{T}_{vv} \,, \tag{40}$$

$$0 = T_{uu} = \mathrm{e}^{-2\phi}(4\rho_{,u}\phi_{,u} - 2\phi_{,uu}) + \hat{T}_{uu} \,, \tag{41}$$

$$0 = T_{uv} = \mathrm{e}^{-2\phi}(2\phi_{,uv} - 4\phi_{,u}\phi_{,v} - \lambda^2\mathrm{e}^{2\rho} + q^2\mathrm{e}^{4\phi+2\rho}) + \hat{T}_{uv} \ . \tag{42}$$

Here, $\hat{T}_{uu}$, $\hat{T}_{vv}$, and $\hat{T}_{uv}$ represent the semiclassical contributions to T_{uu}, T_{vv}, and T_{uv}, respectively, which are given by

$$\hat{T}_{vv} = \frac{1}{2}Z_{,v}^{\,2} - \sqrt{\frac{N}{12}}\rho_{,v}Z_{,v} + \frac{1}{2}\sqrt{\frac{N}{12}}Z_{,vv} + \frac{1}{2}Y_{,v}^{\,2} \,, \tag{43}$$

$$\hat{T}_{uu} = \frac{1}{2}Z_{,u}^{\,2} - \sqrt{\frac{N}{12}}\rho_{,u}Z_{,u} + \frac{1}{2}\sqrt{\frac{N}{12}}Z_{,uu} + \frac{1}{2}Y_{,u}^{\,2} \,, \tag{44}$$

$$\hat{T}_{uv} = -\sqrt{\frac{N}{48}}Z_{,uv} \ . \tag{45}$$

The Maxwell equations also get semiclassical source terms:

$$0 = \frac{8}{g_A^2}\partial_u(\mathrm{e}^{-2(\phi+\rho)}F_{uv}) - \sqrt{2N}\partial_u Y \ , \tag{46}$$

$$0 = \frac{8}{g_A^2}\partial_v(\mathrm{e}^{-2(\phi+\rho)}F_{uv}) - \sqrt{2N}\partial_v Y \ . \tag{47}$$

The dilaton and matter equations of motion (8) and (10) are not modified. [As before, we consider the vacuum solution (11) to Eq. (10).] The variables Z and Y satisfy the field equations

$$0 = -2Z_{,uv} + \sqrt{\frac{N}{3}}\rho_{,uv} \ , \tag{48}$$

$$0 = -2Y_{,uv} - \sqrt{2N}F_{uv} \ . \tag{49}$$

Motivated by the classical relation (13), we define q by

$$\lambda q \equiv (\sqrt{2}/g_A)\ \mathrm{e}^{-2(\phi+\rho)}F_{uv} \,. \tag{50}$$

Note that q is no longer constant: It evolves due to the semiclassical source terms in the modified Maxwell equations (46, 47). These two equations can immediately be integrated:

$$q = q_0 + (g_A/4\lambda)\sqrt{N}\,Y\,, \tag{51}$$

where q_0 is an arbitrary integration constant. The equation for Z can be integrated too:

$$Z = \sqrt{\frac{N}{12}}\rho + Z_v(v) + Z_u(u)\ , \tag{52}$$

where $Z_v(v)$ and $Z_u(u)$ are arbitrary initial functions.

To simplify the equations, we define $K \equiv N/24$, $g \equiv \sqrt{3}\,g_A$, and rescale Y and Z as

$$z = Z/\sqrt{2K}\ ,\ y = Y/\sqrt{2K}\,. \tag{53}$$

Equations (49,51,52) then become, respectively,

$$y_{,uv} = -\sqrt{6}F_{uv}\ , \tag{54}$$

$$q = q_0 + (Kg/\lambda)\,y\,, \tag{55}$$

and

$$z = \rho + z_v(v) + z_u(u)\ , \tag{56}$$

where again $z_v(v)$ and $z_u(u)$ are arbitrary initial functions. The semiclassical contributions to the stress-energy now read

$$\hat{T}_{uu} = K\left(z_{,u}^2 - 2\rho_{,u}z_{,u} + z_{,uu} + y_{,u}^2\right)\ , \tag{57}$$

$$\hat{T}_{vv} = K\left(z_{,v}^2 - 2\rho_{,v}z_{,v} + z_{,vv} + y_{,v}^2\right)\ , \tag{58}$$

$$\hat{T}_{uv} = -K\,\rho_{,uv}\ . \tag{59}$$

Using Eq. (50), we can rewrite the field equation for y in terms of q:

$$y_{,uv} = -\lambda g\ \mathrm{e}^{2(\phi+\rho)}q\ . \tag{60}$$

The closed system of semiclassical field equations is composed of Eq. (17) for the dilaton, the three Einstein equations (40-42), as well as Eqs. (55-60).

Finally, we write the system of semiclassical field equations in the R, S variables. The dilaton equation (17) and the Einstein equation (42) [with Eq. (59)] yield

$$R_{,uv} = \lambda^2 (q^2/R^2 - 1)\ \mathrm{e}^S - K\,\rho_{,uv}\ , \tag{61}$$

$$S_{,uv} = -(2\lambda^2 q^2/R^3)\ \mathrm{e}^S + K\,\rho_{,uv}/R\ , \tag{62}$$

where $\rho = (S - \log R)/2$. The semiclassically-corrected constraint equations take the form

$$R_{,uu} - R_{,u} S_{,u} + \hat{T}_{uu} = 0\ , \tag{63}$$

$$R_{,vv} - R_{,v} S_{,v} + \hat{T}_{vv} = 0\ . \tag{64}$$

Again, this system is supplemented by Eqs. (55 - 60). Equation (60) can be re-expressed using the R, S variables as

$$y_{,uv} = -\lambda g\,(\mathrm{e}^S/R^2)\,q\ . \tag{65}$$

B. Semiclassical fluxes outside the black hole

We turn now to analyze the evolution of the quantum variables Y and Z, in order to obtain the semiclassical fluxes. To that end we use the adiabatic approximation. Namely, we view Y and Z as test fields living on the background described by the static classical solution (with fixed m and q).

We first calculate the semiclassical electric currents outside the BH. Using Eq. (27), we rewrite Eq. (65) as

$$y_{,uv} = -\lambda g q H/R^2\ . \tag{66}$$

We now integrate this equation with respect to v, recalling $dv = dx = dR/H$:

$$y_{,u} = -\lambda gq \int (H/R^2)dv = -\lambda gq \int R^{-2}dR = \lambda gq/R + J_u(u)\ . \tag{67}$$

Similarly, we find for y_v (recalling $du = -dx$):

$$y_{,v} = -\lambda gq \int (H/R^2)du = \lambda gq \int R^{-2}dR = -\lambda gq/R + J_v(v)\ . \tag{68}$$

The integration constants, i.e. the functions $J_u(u)$ and $J_v(v)$, are to be determined from the initial conditions. Since we assume no ingoing current is coming from past null infinity, we must set $J_v(v) = 0$. Also, regularity at the EH requires that $y_{,u}$ vanishes there, which implies $J_u(u) = -\lambda gq/R_+$. Therefore,

$$y_{,u} = \lambda gq(1/R - 1/R_+) \quad , \quad y_{,v} = -\lambda gq/R\,, \tag{69}$$

and from Eq. (55) we obtain

$$q_{,u} = Kg^2 q(1/R - 1/R_+) \quad , \quad q_{,v} = -Kg^2 q/R\,. \tag{70}$$

The fluxes $\hat{T}_{vv}$ and $\hat{T}_{uu}$, Eqs. (57,58), can be expressed explicitly by means of Eq. (56):

$$\hat{T}_{uu} = K[(\rho_{,uu} - \rho_{,u}^2) + y_{,u}^2 + \hat{z}_u(u)]\,, \tag{71}$$

$$\hat{T}_{vv} = K[(\rho_{,vv} - \rho_{,v}^2) + y_{,v}^2 + \hat{z}_v(v)]\,, \tag{72}$$

where $\hat{z}_v(v) \equiv z_{v,vv} + z_{v,v}^2$ and $\hat{z}_u(u) \equiv z_{u,uu} + z_{u,u}^2$. Substituting Eq. (69), we find

$$\hat{T}_{uu} = K[(\rho_{,uu} - \rho_{,u}^2) + \lambda^2 g^2 q^2 (1/R - 1/R_+)^2 + \hat{z}_u(u)]\,, \tag{73}$$

$$\hat{T}_{vv} = K[(\rho_{,vv} - \rho_{,v}^2) + \lambda^2 g^2 q^2/R^2 + \hat{z}_v(v)]\,. \tag{74}$$

The functions $\hat{z}_v(v)$ and $\hat{z}_u(u)$ are to be chosen such that no influx is coming from past null infinity, and the outflow is regular at $R = R_+$, that is,

$$\hat{T}_{vv}(R = \infty) = 0 \quad , \quad \hat{T}_{uu}(R = R_+) = 0\,. \tag{75}$$

In the static classical background we have $\rho = (1/2)\log(H/R)$, so

$$-\rho_{,u} = \rho_{,v} = \rho_{,x} = H\rho_{,R} = (R/2)(H/R)_{,R} \tag{76}$$

and

$$\rho_{,uu} = \rho_{,vv} = \rho_{,xx} = H[(R/2)(H/R)_{,R}]_{,R} \tag{77}$$

Note that $\rho_{,uu}$ and $\rho_{,vv}$ (as well as $\rho_{,uv}$) vanish both at R_+ and at $R = \infty$. On the other hand, $\rho_{,u}$ and $\rho_{,v}$ vanish at $R = \infty$, but at $R = R_+$ they get a finite value,

$$-\rho_{,u} = \rho_{,v} = \kappa_+ \,, \qquad (R = R_+) \tag{78}$$

In order for $\hat{T}_{vv}$ to vanish at $R = \infty$, we choose $\hat{z}_v = 0$ and obtain

$$\hat{T}_{vv} = K[(\rho_{,vv} - \rho_{,v}^2) + (\lambda g q/R)^2\,] \,. \tag{79}$$

Also, the demand that $\hat{T}_{uu}$ vanishes at $R = R_+$ yields $\hat{z}_u = \kappa_+^2$, namely,

$$\hat{T}_{uu} = K\left[\kappa_+^2 + (\rho_{,uu} - \rho_{,u}^2) + [\lambda g q(1/R - 1/R_+)]^2\right] \,. \tag{80}$$

C. Semiclassical fluxes inside the black hole

Before we discuss semiclassical effects inside the BH, we need to extend our classical solution for the static black-hole background to the internal region. Clearly, Eq. (27) as it stands is not valid at $R_- < R < R_+$, where H is negative. The internal solution in double null, Eddington-like coordinates is obtained form Eq. (27) by minor changes of sign. The main difference is that, inside the BH the variable x (the only variable on which the solution depends) is *temporal* rather than spatial, namely, $x = v + u$. The internal solution takes the form

$$\mathrm{e}^S = -H(R) \;,\; R_{,x} = H(R) \;,\; x = v + u \qquad (R_- < R < R_+) \,. \tag{81}$$

Correspondingly, the metric function g_{uv} is given by

$$-2g_{uv} = e^{2\rho} = -H/R \,.$$

We can now repeat the calculations of the semiclassical fluxes. The initial conditions for the outgoing fluxes at $v \to -\infty$ are the same as in the external problem: Both the energy and charge outfluxes must vanish at $R = R_+$, in order to achieve regularity at the EH. The initial conditions for the *ingoing* fluxes at $u \to -\infty$ (the EH) are dictated by continuity: At the EH, both $q_{,v}$ and $\hat{T}_{vv}$ must continuously match the corresponding quantities in the external region $R > R_+$ (recall that the coordinate v is regular at the EH).

The calculation now proceeds in a way completely analogous to the external semiclassical calculations of the previous subsection, except for a few changes of sign. For example, when calculating the charge fluxes, one must recall that $\mathrm{e}^S = -H$ and $du = dx$, and as a consequence $q_{,u}$ changes sign (but not $q_{,v}$):

$$q_{,u} = -Kg^2 q(1/R - 1/R_+) \quad , \quad q_{,v} = -Kg^2 q/R \qquad (R_- < R < R_+)\,. \tag{82}$$

The energy-momentum fluxes are given by Eqs. (79) and (80) without any change (recall, though, that now ρ is given by $e^{2\rho} = -H/R$).

IV. EVAPORATION OF THE BLACK HOLE

A. Evolution of m and q

In order to calculate the rate of change of m and q, as measured by an observer at future null infinity (FNI), we need to evaluate the outgoing fluxes at the limit $R \to \infty$. For brevity we denote the u -derivatives of m and q at FNI by an overdot. At this limit Eq. (70) reads

$$\dot{q} = -Kg^2\, q/R_+\,. \tag{83}$$

Taking the large-R limit in Eq. (80), we find

$$\hat{T}_{uu} = K[\kappa_+^2 + (\lambda g q/R_+)^2] = K\lambda^2[\lambda^2(m^2 - q^2) + g^2 q^2]/R_+^2 \qquad (\mathrm{FNI})\,. \tag{84}$$

The relation between $\dot{m}$ and $\hat{T}_{uu}$ is most easily expressed in terms of the BH's physical mass $M = 2\lambda m$ and the asymptotically-flat null coordinate $\tilde{u} = \lambda u$ (see section II): The change in the Bondi mass M is simply (minus) the integral of $\hat{T}_{\tilde{u}\tilde{u}}$ with respect to $\tilde{u}$ along FNI; that is,

$$\partial M/\partial\tilde{u} = -\hat{T}_{\tilde{u}\tilde{u}} = -\lambda^{-2}\hat{T}_{uu} \qquad \text{(FNI)}\,.$$

We find that

$$\dot{m} = -\hat{T}_{uu}/2\lambda^2 \qquad \text{(FNI)}\,. \tag{85}$$

Alternatively, we can derive this relation using the Vaidya-like solution constructed in Appendix A. To that end, we recall that the ingoing fluxes of both energy and charge, as well as the semiclassical correction term $\hat{T}_{uv}$, vanish at FNI (where $R \to \infty$) – cf. Eqs. (69, 79). The only semiclassical correction terms which survive at FNI are the Hawking energy outflux $\hat{T}_{uu}$ and charge outflux $q_{,u}$. We can therefore represent the solution near FNI by the (charged) outgoing Vaidya-like solution. This exact solution provides the desired relation between $\dot{m}$ and the energy outflux at FNI. Identifying $\hat{T}_{uu}$ with $T^{(flux)}_{uu}$ in Eq. (A10) below, we recover the relation (85).

Substituting the above expression for $\hat{T}_{uu}$(FNI) in Eq. (85), we find

$$\dot{m} = -K[\kappa_+^2/\lambda^2 + (gq/R_+)^2]/2 = -K[\lambda^2(m^2-q^2) + g^2q^2]/2R_+^2\,. \tag{86}$$

To verify the consistency of the above results for $\dot{m}$ and $\dot{q}$ (and, more generally, for the fluxes of energy-momentum and charge), we can calculate the rate of change of R_+ in two different ways. First, since the evaporation is very slow (corresponding to a large, macroscopic, BH), we can view the geometry as quasi-static. At each "moment" u, R_+ (as viewed by a distant observer) can be estimated by the momentary values of m and q, via the standard, static-solution relation $R_+ = m + (m^2 - q^2)^{1/2}$. Alternatively, we can apply the constraint equation $R_{,vv} - R_{,v}S_{,v} + \hat{T}_{vv} = 0$ to the null generators of the EH, and in this way to analyze the rate of contraction of R_+. Since the evolution is slow (and hence, on time scales short compared to the BH's evaporation time, the geometry only depends on $x = v - u$ to the leading order), $\partial R_+/\partial v$ at the EH must coincide with $\dot{R}_+ \equiv \partial R_+(m,q)/\partial u$ at FNI, i.e. with $\dot{m}\,R_{+,m} + \dot{q}\,R_{+,q}$. In Appendix B we calculate these two quantities and show they are indeed the same:

$$R_{+,v}(\mathrm{EH}) = \dot{\mathrm{R}}_+ = -\mathrm{K}\lambda^2[\lambda^2(\mathrm{m}^2 - \mathrm{q}^2) - \mathrm{g}^2\mathrm{q}^2\,]/(2\kappa_+\mathrm{R}_+^2)\,. \tag{87}$$

Finally, let us compare our result (84) for the Hawking outflux at FNI to the standard result obtained by CGHS (in the uncharged case). Taking the limit $q = 0$ in Eq. (84), one finds

$$\hat{T}_{uu} = K\lambda^4 m^2/R_+^2 = K\lambda^4/4 \qquad\qquad (\mathrm{FNI}\,,\ \mathrm{q}=0)\,. \tag{88}$$

Transforming this result to the asymptotically-flat $\tilde{u}, \tilde{v}$ coordinates defined in section II (which is also the gauge used by CGHS), we find

$$\hat{T}_{\tilde{u}\tilde{u}} = K\lambda^2/4 = N\lambda^2/96 \qquad\qquad (\mathrm{FNI}\,,\ \mathrm{q}=0)\,. \tag{89}$$

This is just one half of the Hawking outflux in the CGHS model. This difference is because the quantum matter field used in the present model is fermionic, whereas that used in the CGHS model is bosonic [10].

B. Evolution of the charge-to-mass ratio

Equations (83) and (86) form a closed autonomous system, which allows us to analyze the evolution of the charge-to-mass ratio upon evaporation. One finds

$$dm/dq = (R_+/2q)\left[(\kappa_+/\lambda g)^2 + q^2/R_+^2\right]\,. \tag{90}$$

Since the right-hand side only depends on q and m through q/m, this equation admits solutions of the form

$$m = cq\,, \tag{91}$$

where c is a positive constant [to be determined from an algebraic equation based on Eq.(90), as we shortly show]. We shall refer to a solution of this form as the *linear solution*. To analyze this solution, we rewrite Eq. (90) as

$$dm/dq = (R_+/2q)\left[(\kappa_+/\lambda g)^2 - 1\right] + m/q\,. \tag{92}$$

This form makes it obvious that for any $0 < g < \lambda/2$, there exists a linear solution of the form (91), with c defined by the algebraic equation $\kappa_+(m = cq) = \lambda g$, i.e.

$$g/\lambda = 1 - \left(1 + \sqrt{1 - c^{-2}}\right)^{-1} \tag{93}$$

[cf. Eq. (32)]. Explicitly we find

$$c = \left(1 - [(1 - g/\lambda)^{-1} - 1]^2\right)^{-1/2} \qquad (0 < g < \lambda/2)\,. \tag{94}$$

Next we analyze the stability of the linear solution (91, 94). To that end, we define $\delta \equiv m/q - c$, and write the evolution equation for δ in the form

$$\frac{d\delta}{d\ln q} = \frac{dm}{dq} - \frac{m}{q} = (R_+/2q)\left[(\kappa_+/\lambda g)^2 - 1\right] \equiv \Gamma(\delta)\,. \tag{95}$$

Note that κ_+ is an increasing function of δ [cf. Eq. (32)]. Since $\lambda g = \kappa_+(m/q = c) = \kappa_+(\delta = 0)$, the term in squared brackets is an increasing function of δ which vanishes for $\delta = 0$. Therefore, Γ has the same sign as δ , which means that $|\delta|$ is an increasing function of q. This implies that upon evaporation (q decreases), $|\delta|$ decreases. Namely, the linear solution (91, 94) is stable. Moreover, since the only zero of Γ is at $\delta = 0$, the linear solution is in fact a global atractor for any $0 < g < \lambda/2$, provided that initially $q > 0$. [For small δ, we can linearize Γ by $\Gamma \cong \beta\delta$, with some constant $\beta = \beta(g) > 0$. We then find that $\delta \propto q^\beta \propto m^\beta$.]

The dynamical system (83,86) has another, trivial, solution:

$$q/m = 0\,. \tag{96}$$

For $0 < g < \lambda/2$, this solution must be unstable: As we have just found, the linear solution (91,94) is a global atractor in this range for any initial $q > 0$. However, for $g > \lambda/2$, for which the above linear solution does not exist (c is not real), the trivial solution $q = 0$ becomes stable. To verify this, we define in this case $\delta \equiv m/q > 0$, and analyze Eq. (95). Since now $\kappa_+/\lambda g < 1$ (for any q/m), $\Gamma(\delta)$ is always negative, meaning that upon evaporation δ increases monotonically. Furthermore, since $R_+/m \geq 1$, the quantity $d\ln\delta/d\ln q = \Gamma/\delta$

is bounded above by the strictly negative number $\gamma/2$, where $\gamma \equiv \lambda^2/4g^2 - 1 < 0$. This means that upon evaporation ($\ln q \to -\infty$), δ gets unboundedly large positive values. Once δ becomes large, we can use the linear approximation $\Gamma/\delta \cong \gamma$ (obtained by approximating $R_+ \cong 2m$ and $\kappa_+ \cong \lambda^2/2$), which yields $\delta \propto q^\gamma$. This implies $m \propto q^{\lambda^2/4g^2}$, namely,

$$q/m \propto m^{4g^2/\lambda^2-1} \qquad (g > \lambda/2)\,. \tag{97}$$

It should be pointed out that this linear analysis of solutions with $q/m << 1$, and particularly the result (97), applies to *any* g. It indicates the stability of the solution $q/m = 0$ in the range $g > \lambda/2$, and its instability in the range $g < \lambda/2$. Thus, for $g > \lambda/2$, Eq. (97) is realized as the late-time, stable, asymptotic behavior. For $g < \lambda/2$, however, even if initially $q/m << 1$, upon evaporation it grows according to Eq. (97) until the linear approximation breaks (provided, of course, that initially $q > 0$). Subsequently q/m converges to a nonzero value c^{-1}, as was discussed above.

We conclude that for $g < \lambda/2$, the charge-to mass ratio converges to a nonzero value,

$$\frac{q}{m} \to \sqrt{1 - [(1 - g/\lambda)^{-1} - 1]^2} \qquad (0 < g < \lambda/2)\,. \tag{98}$$

This value is independent of the initial values of q and m (though it only holds if q is initially nonzero). However, for $g > \lambda/2$, the charge-to mass ratio decreases as a power of m, and eventually approaches zero, as described in Eq. (97).

V. SUMMARY

We presented here a dilatonic two-dimensional model of a charged black hole. On the classical level, our model yields static charged BHs, characterized by the two parameters m and q (representing the BH's mass and charge). These static BHs admit an inner horizon instead of a spacelike singularity. Then we studied the semiclassical effects (on the background of the above classical, static, BH solution), using the method developed in Ref. [6] . We derived explicit expressions for the fluxes of charge and energy-momentum as a function of the "radius" R, both outside and inside the BH.

By analyzing the outflux of energy-momentum and charge at future null infinity (and also the influx at the EH), we calculated the evaporation rate of both m and q, as a function of these two quantities. This yields a system of two coupled first-order differential equations, i.e. $\dot{q}(m,q)$ and $\dot{m}(m,q)$ [Eqs. (83) and (86), respectively]. We then analyzed the evolution of the ratio q/m upon evaporation. Depending on the value of the electromagnetic coupling constant g (recall $g \equiv \sqrt{3}\, g_A$), there are two different regimes: For $g > \lambda/2$, upon evaporation q/m decays to zero as described in Eq. (97) above. On the other hand, for $g < \lambda/2$, the charge-to-mass ratio approaches a non-zero constant given in Eq. (98). This constant depends on g but not on the initial values of m and q (provided that q is initially nonvanishing). Note that this final charge-to-mass ratio approaches extremality ($q/m = 1$) at the limit $g \to 0$, and $q/m \to 0$ at the limit $g \to \lambda/2$.

As was explained in the Introduction, no attempt was made here to investigated the final state of evaporation. The analysis throughout this paper was restricted to the macroscopic phase, i.e. to the stage where the mass is sufficiently large. This restriction is necessary for the validity of the adiabatic approximation: This approximation assumes that in evaluating the semiclassical fluxes (more specifically, when solving the field equations for the quantum variables Y and Z), m and q may be regarded as fixed parameters (and the background geometry may be approximated by the corresponding static BH solution). Clearly, this approximation is only valid as long as the change in m during a dynamical time scale is much smaller than m itself. The dynamical time scale (expressed in terms of u and/or v) is of order $1/\kappa_+$, which is typically of order $\sim \lambda^{-2}$. The mass evaporation rate $\dot{m}$ is of order $K\lambda^2$ (recall $K \equiv N/24$). Thus, the macroscopic phase – the domain of validity of the adiabatic approximation – is given by

$$m \gg K \,. \tag{99}$$

In this domain the dilaton field ϕ outside the BH satisfies

$$e^{-2\phi} \gg K \,.$$

(This also holds inside the BH, in the region $R > R_-$ – provided that q/m is not too small.) This is known to be the "weak-coupling" domain in large-N dilaton gravity. It should also be pointed out that the curvature singularity found in Ref. [2] (in the uncharged case) occurs at $e^{-2\phi} = 2K$, which does not occur in the macroscopic domain considered here.

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APPENDIX A: VAIDYA-LIKE SOLUTIONS

In this Appendix we describe the construction of the Vaidya-like solution in double-null coordinates. For concreteness we shall consider here the outgoing solution, which is useful for describing the geometry of an evaporating BH (in the weak-field region), but the ingoing solution can be constructed in a completely analogous manner.

Consider first the case of an uncharged null fluid (i.e. q is a fixed parameter). The solution is uniquely determined by the function $m(u)$. In analogy with the construction of the static vacuum-like solutions in section II, we define

$$H(R,u) \equiv \lambda^2 \left[R - 2m(u) + q^2/R \right] \,. \tag{A1}$$

The function $R(u,v)$ is now determined by the ordinary differential equation

$$R_{,u} = -H(R,u) \,. \tag{A2}$$

This equation is to be integrated along the lines of constant v (with each ingoing null line having its own "initial value" for R), and this integration produces the function $R(u, v)$.[3] Then, $S(u, v)$ is given by

$$e^S = R_{,v} \,. \tag{A3}$$

Differentiating Eq. (A2) with respect to v, one recovers the field equation (23) for R, i.e. $R_{,uv} = Fe^S$. Next, differentiating Eq. (A3) with respect to u, one obtains $S_{,u} = F$, and a second differentiation with respect to v now yields the field equation for S.

To discuss the null-fluids content of the Vaidya-like solutions (or, more generally, the radiative solutions), it is useful to re-write the constraint equations as

$$0 = T_{uu} = R_{,uu} - R_{,u}S_{,u} + T_{uu}^{(flux)} \,, \tag{A4}$$

$$0 = T_{vv} = R_{,vv} - R_{,v}S_{,v} + T_{vv}^{(flux)} \,. \tag{A5}$$

The components $T_{uu}^{(flux)}$ and $T_{vv}^{(flux)}$ then describe the energy-momentum carried by the outgoing and ingoing null fluids, respectively. (The vacuum-like solutions then correspond to $T_{uu}^{(flux)} = T_{vv}^{(flux)} = 0$.) Differentiating Eq. (A2) with respect to v yields $R_{,v}S_{,v} = R_{,vv}$. Namely, the outgoing Vaidya-like solution is characterized by a vanishing influx:

$$T_{vv}^{(flux)} = 0 \,. \tag{A6}$$

To express the flux in the u direction in terms of $m(u)$, we notice that the above result $S_{,u} = F$ together with Eq. (A2) yield $R_{,u}S_{,u} = -FH$ and $R_{,uu} = -FH - \partial H/\partial u$ [where $\partial H/\partial u \equiv (\partial H/\partial m)(\partial m/\partial u)$], and hence

$$T_{uu}^{(flux)} = R_{,u}S_{,u} - R_{,uu} = \partial H/\partial u = -2\lambda^2 m_{,u} \,. \tag{A7}$$

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[3]This construction fixes the gauge for the outgoing coordinate u, but leaves the gauge for the ingoing coordinate v uncpecified. In this construction v enters as a parameter which parametrizes the one- parameter set of solutions to the differential equation (A2) for R. (For example, one can take v to be the value of R on some "initial" outgoing ray $u = u_0$.)

Note that $T_{uu}^{(flux)}$ is independent of v – this is an important feature of the Vaidya-like solutions [valid for any $F(R)$].

Strictly speaking, the field equations in the form (23,24) assume that q is a constant. One can, however, immediately generalize it by allowing q to be a function of u and v. When considering the (outgoing) Vaidya-like solutions, it is most natural to assume that q (like m) depends on u only. Physically, this would correspond to a model with an outflow of charged null fluid. This generalization is important because in our model the Hawking outflux is indeed charged.

The generalization of the Vaidya-like solution to the charged null fluid case is straightforward. One simply replaces Eq. (A1) by

$$H(R,u) \equiv \lambda^2 \left[R - 2m(u) + q(u)^2/R\right] . \tag{A8}$$

The rest of the above construction, Eqs. (A2) and (A3), are unchanged. The solution is uniquely determined by the two functions $m(u)$ and $q(u)$, which describe the outflux of mass and charge, respectively. One can verify that the evolution equations (19,20) are satisfied, as well as Eq. (A6). However, the energy-momentum content of the outgoing flux is now modified:

$$T_{uu}^{(flux)} = R_{,u}S_{,u} - R_{,uu} = \partial H/\partial u = -2\lambda^2 \left(m_{,u} - q\, q_{,u}/R\right) . \tag{A9}$$

Note that $T_{uu}^{(flux)}$ is no longer constant along lines of constant u . This has a simple physical interpretation: The Lorentz force acting on the charged outflux does a work on it, and changes its energy-momentum content. (This situation is well known in the context of the four-dimensional, spherically symmetric, charged Vaidya solution [11]; see the discussion in [12] and [13].) Note, however, that at FNI the term $q_{,u}/R$ vanishes, and one again obtains

$$T_{uu}^{(flux)} = -2\lambda^2 m_{,u} \qquad \text{(FNI)} . \tag{A10}$$

We can verify this result by relating $T_{uu}^{(flux)}$ to the rate of change of the Bondi mass M. In terms of the asymptotically-flat null coordinate $\tilde{u}$, these two quantities are related by

$$\partial M/\partial\tilde{u} = -\hat{T}_{\tilde{u}\tilde{u}}\,.$$

Substituting $M = 2\lambda m$, $\tilde{u} = \lambda u$, and $\hat{T}_{\tilde{u}\tilde{u}} = \lambda^{-2}\hat{T}_{uu}$, we find $2m_{,u} = \partial M/\partial\tilde{u} = -\lambda^{-2}\hat{T}_{uu}$, which conforms with Eq. (A10).

The Vaidya-like solution may be interpreted as a slowly varying, quasi-static solution, which is described by the vacuum-like solution (27) – except that the BH's mass and charge are slowly evaporating. This interpretation is meaningful as long as the evaporating BH is macroscopic (i.e. the evaporation time scale is much larger than the dynamical time scale). Note that in this quasi-static limit the coordinate u used in the above construction of the Vaidya-like solution coincides with the Eddington-like coordinate u of the static, vacuum-like, solution (27). That the gauges of these two solutions agree can be seen, for example, by recognizing that $R_{,u} = -H$ in both solutions.

APPENDIX B: RATE OF CONTRACTION OF THE EVENT HORIZON

In this Appendix we shall calculate the contraction rate of R_+ in two different ways, as outlined in section IV. The equality of the two results may serve as a consistency test for the expressions derived above for $\dot{m}$ and $\dot{q}$.

First, since the evaporation is slow, we may assume that at each moment u, R_+ is given by

$$R_+(u) = m(u) + [m(u)^2 - q(u)^2]^{1/2}\,.$$

Taking u-derivatives of all quantities at FNI, one finds

$$\dot{R}_+ = \dot{m}\left[1 + m/(m^2 - q^2)^{1/2}\right] - \dot{q}q/(m^2 - q^2)^{1/2}\,. \tag{B1}$$

Noting that

$$1 + m/(m^2 - q^2)^{1/2} = R_+/(m^2 - q^2)^{1/2} = \lambda^2/\kappa_+$$

[cf. Eq. (31)], we can re-write Eq. (B1) as

$$\kappa_+ \dot{R}_+ = \lambda^2 (\dot{m} - \dot{q} q / R_+) \, .$$

Substituting the above expressions for $\dot{m}$ and $\dot{q}$, one finds

$$\kappa_+ \dot{R}_+ = -K\lambda^2 [\lambda^2 (m^2 - q^2) - g^2 q^2] / 2R_+^2 \, . \tag{B2}$$

In the second way, we apply the constraint equation (64) to the null generators of the EH, and use it to analyze the rate of decrease of R_+ with v. Since the evolution is assumed to be slow, we can neglect the term $R_{,vv}$ and write

$$S_{,v} \, R_{+,v} = \hat{T}_{vv} \qquad (EH) \, .$$

Using the background solution (27) one can easily verify that at the EH $S_{,v} = dH/dR = 2\kappa_+$, and Eq. (79) yields

$$\hat{T}_{vv} = -K [\kappa_+^2 - (\lambda g q / R_+)^2 \,] \qquad (EH) \, . \tag{B3}$$

Therefore,

$$\kappa_+ \, R_{+,v} = \hat{T}_{vv}/2 = -K\lambda^2 [\lambda^2 (m^2 - q^2) - g^2 q^2 \,] / 2R_+^2 \qquad (EH) \, . \tag{B4}$$

Comparing Eqs. (B2) and (B4), we find that indeed $R_{+,v}$ at the EH is exactly the same as $\dot{R}_+$, as one should expect.

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